§7.3 Improper Integrals

In this paragraph we consider integration domains which are infinite or finite integration domains where the integrand is not bounded. In some of these cases the integral still can be defined and is called an "improper integral".

Case I: one of the integration boundaries is infinite.

Definition 7.8:

Let $f: [a, \infty) \longrightarrow \mathbb{R}$ be a function, which is integrable over the interval $[a, R]$, $a < R < \infty$. If the limit

$$\lim_{R \to \infty} \int_a^R f(x)\,dx$$

exists, the integral $\int_a^\infty f(x)\,dx$ is called convergent and one sets

$$\int_a^\infty f(x)\,dx := \lim_{R \to \infty} \int_a^R f(x)\,dx.$$

Analogously, one defines the integral $\int_{-\infty}^{a} f(x)dx$ for a function $f: (-\infty, a] \longrightarrow \mathbb{R}$.

<u>Example 7.10:</u>

The integral $\int_1^\infty \frac{dx}{x^s}$ converges for $s > 1$.

We have

$$\int_1^R \frac{dx}{x^s} = \frac{1}{1-s} \cdot \frac{1}{x^{s-1}} \Big|_1^R = \frac{1}{s-1} \left(1 - \frac{1}{R^{s-1}}\right)$$

As $\lim_{R \to \infty} \frac{1}{R^{s-1}} = 0$, we have

$$\int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1} \quad \text{for } s > 1.$$

On the other hand: $\int_1^\infty \frac{dx}{x^s}$ does <u>not</u> converge for $s \leq 1$. For example, for $s = 1$:

$$\int_1^R \frac{dx}{x} = \log R \longrightarrow \infty \quad (R \to \infty).$$

<u>Case II:</u> The integrand is not defined at one of the integration borders.

<u>Definition 7.9:</u>

Let $f: (a, b] \longrightarrow \mathbb{R}$ be a function, which is

Riemann-integrable over every interval $[a+\varepsilon, b]$, $0 < \varepsilon < b-a$. If the limit

$$\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx$$

exists, the integral $\int_a^b f(x)\,dx$ is called convergent and one sets

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx .$$

<u>Example 7.11:</u>

The integral $\int_0^1 \frac{dx}{x^s}$ converges for $s<1$. We have

$$\int_\varepsilon^1 \frac{dx}{x^s} = \frac{1}{1-s} \cdot \frac{1}{x^{s-1}} \Big|_\varepsilon^1 = \frac{1}{1-s}\left(1-\varepsilon^{1-s}\right).$$

As $\lim_{\varepsilon \searrow 0} \varepsilon^{1-s} = 0$, we get

$$\int_0^1 \frac{dx}{x^s} = \frac{1}{1-s} \quad \text{for } s<1$$

On the other hand, one shows

$$\int_0^1 \frac{dx}{x^s} \text{ does \underline{not} converge for } s \geq 1.$$

<u>Case III:</u> Both integration borders are critical.

Definition 7.10:

Let $f: (a,b) \longrightarrow \mathbb{R}$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$, be Riemann-integrable over every compact sub-interval $[\alpha, \beta] \subset (a,b)$ and let $c \in (a,b)$ be arbitrary. If the two improper integrals

$$\int_a^c f(x)\,dx = \lim_{\alpha \searrow a} \int_\alpha^c f(x)\,dx$$

and

$$\int_c^b f(x)\,dx = \lim_{\beta \nearrow b} \int_c^\beta f(x)\,dx$$

converge, the integral $\int_a^b f(x)\,dx$ is called convergent and one sets

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note: This definition is independent from the choice of $c \in (a,b)$.

Example 7.12:

i) The integral $\int_0^\infty \frac{dx}{x^s}$ diverges for every $s \in \mathbb{R}$.

ii) The integral $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ converges:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \searrow 0} \int_{-1+\varepsilon}^{0} \frac{dx}{\sqrt{1-x^2}} + \lim_{\varepsilon \searrow 0} \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}}$$

$$= -\lim_{\varepsilon \searrow 0} \arcsin(-1+\varepsilon) + \lim_{\varepsilon \searrow 0} \arcsin(1-\varepsilon)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.$$

iii) The integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ also converges:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R \to \infty} \int_{R}^{0} \frac{dx}{1+x^2} + \lim_{R \to \infty} \int_{0}^{R} \frac{dx}{1+x^2}$$

$$= -\lim_{R \to \infty} \arctan(-R) + \lim_{R \to \infty} \arctan(R)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

Proposition 7.16:

Let $f : [1, \infty) \to \mathbb{R}_+$ be a monotonically decreasing function. Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_{1}^{\infty} f(x)\,dx \text{ converges.}$$

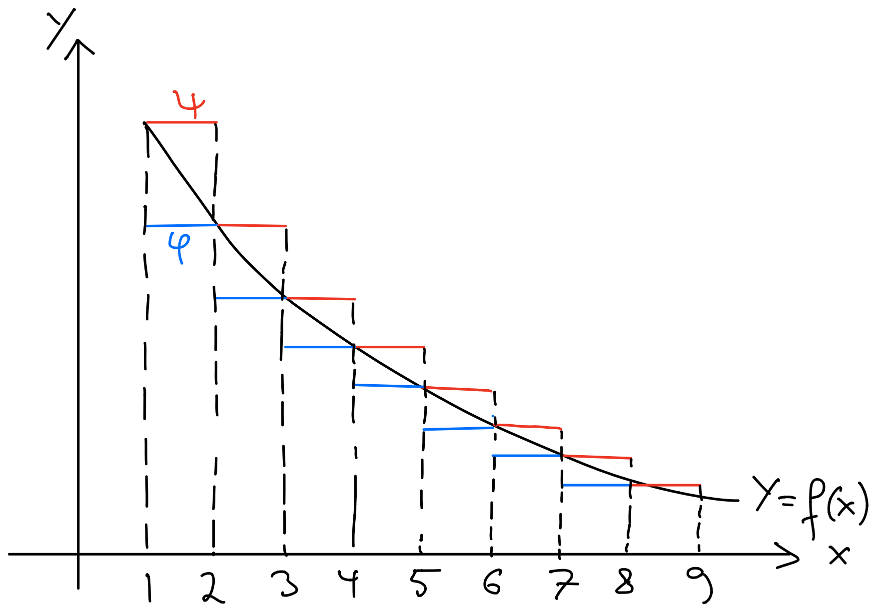
Proof:

We define step functions $\varphi, \psi : [1,\infty) \to \mathbb{R}$ through

$$\left.\begin{aligned} \psi(x) &:= f(n) \\ \varphi(x) &:= f(n+1) \end{aligned}\right\} \text{ for } n \le x < n+1.$$

As f is monotonically decreasing, we have

$$\varphi \le f \le \psi.$$



Integrating over the interval $[1, N]$ then gives:

$$\sum_{n=2}^{N} f(n) = \int_1^N \varphi(x)\,dx \le \int_1^N f(x)\,dx \le \int_1^N \psi(x)\,dx = \sum_{n=1}^{N-1} f(n).$$

If $\int_1^\infty f(x)\,dx$ converges, the sequence $\sum_{n=1}^\infty f(n)$ is bounded, so convergent. On the other hand, if $\sum_{n=1}^\infty f(n)$ is a convergent sequence, then it follows that $\int_1^R f(x)\,dx$ is monotonically increasing and bounded for $R \to \infty$, thus bounded.

<u>Example 7.13:</u>

i) From Ex. 7.10, namely the convergence of $\int_1^\infty \frac{dx}{x^s}$ for $s>1$, it follows that the sequence $\sum_{n=1}^\infty \frac{1}{n^s}$ is convergent for $s>1$ and divergent for $s \leq 1$.

This way, one obtains

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s}, \quad (s>1)$$

which is called "Riemann's zeta-function".

ii) As $\int_1^N \frac{dx}{x} = \log N$, the sum $\sum_{n=1}^N \frac{1}{n}$ grows approximately as fast (or slow) as $\log N$ to ∞. More precisely: there exists a constant $\gamma \in [0,1]$, s.t.

$$\gamma = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$$

<u>Proof:</u>

Prop. 7.16 gives for $N > 1$

$$\sum_{n=2}^{N} \frac{1}{n} \leq \int_1^N \frac{dx}{x} = \log N \leq \sum_{n=1}^{N-1} \frac{1}{n}.$$

$$\Rightarrow \quad \frac{1}{N} \leq \gamma_N := \sum_{n=1}^{N} \frac{1}{n} - \log N \leq 1.$$

and

$$\gamma_{N-1} - \gamma_N = \int_{N-1}^{N} \frac{dx}{x} - \frac{1}{N} = \int_{N-1}^{N} \left(\frac{1}{x} - \frac{1}{N} \right) dx > 0$$

$\Rightarrow$ γ_N is monotonically decreasing and bounded from below by 0.

Thus the limit

$$\gamma = \lim_{N \to \infty} \gamma_N = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$$

exists.

$\square$

Remark 7.7:

One can rewrite the above also as follows:

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O(1) \quad \text{for } N \to \infty$$

The number γ is called "Euler-Mascheroni" constant and its numerical value is

$$\gamma = 0.57721566\ldots$$

It is not known whether γ is rational, irrational or even transcendental.